The Grad-Shafranov Equations of Axisymmetric, Nonstationary Models of the Central Engine in an Active Galactic Nucleus

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II. BLACK HOLE ASTROPHYSICS

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INTRODUCTION
1. INTRODUCTION

Black Hole Astrophysics
→ Black Hole Thermodynamics
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→ Black Hole Magnetospheres & Jets
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II
BLACK HOLE ASTROPHYSICS
In cylindrical coordinates $(\varpi, \varphi, z)$,

Electric Field \[ \mathbf{E} = E_\varpi \mathbf{e}_\varpi + E_\varphi \mathbf{e}_\varphi + E_z \mathbf{e}_z \]

Magnetic Field \[ \mathbf{B} = B_\varpi \mathbf{e}_\varpi + B_\varphi \mathbf{e}_\varphi + B_z \mathbf{e}_z \]

Toroidal Components : \[ \mathbf{E}^T = E_\varphi \mathbf{e}_\varphi \text{ and } \mathbf{B}^T = B_\varphi \mathbf{e}_\varphi \]

Poloidal Components : \[ \mathbf{E}^P = E_\varpi \mathbf{e}_\varpi + E_z \mathbf{e}_z \text{ and } \mathbf{B}^P = B_\varpi \mathbf{e}_\varpi + B_z \mathbf{e}_z \]

where $\varpi$ is the perpendicular distance separating the symmetry axis of an arbitrary FIDO (Fiducial Observer) and $(\mathbf{e}_\varpi, \mathbf{e}_\varphi, \mathbf{e}_z)$ are orthonormal unit vectors. (In Newtonian case, $\varpi \to R$)
II. BLACK HOLE ASTROPHYSICS II

In cylindrical coordinates \((\varpi, \varphi, z)\),

Axisymmetric Conditions: \(\frac{\partial f}{\partial \varphi} = 0\) and \(\frac{\partial f}{\partial \varphi} = 0\)

Stationary Conditions: \(\frac{\partial f}{\partial t} = 0\) and \(\frac{\partial f}{\partial t} = 0\)

Nonstationary Conditions: \(\frac{\partial f}{\partial t} \neq 0\) and \(\frac{\partial f}{\partial t} \neq 0\)

\(f\) and \(\mathbf{f}\): any scalar and vector, resp.
We assume that the following highly conducting plasma condition is satisfied everywhere:

\[ \mathbf{E} + \frac{1}{c} \mathbf{v}_F \times \mathbf{B} \simeq 0, \quad (2.1) \]

while the force-free condition

\[ \rho_e \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \simeq 0 \quad (2.2) \]

is satisfied only in Region 1 (i.e., Relativistic).
The cross section of an active galactic nucleus (AGN).
II. BLACK HOLE ASTROPHYSICS V

(Source: Internet Encyclopedia of Science)
Let us consider a FIDO (Fiducial Observer) that is fixed at a given point near a Kerr black hole. In this case, Maxwell’s equations, which satisfy the axisymmetric condition, can be written as follows:

\[
\nabla \cdot \mathbf{E} = 4\pi \rho_e, \tag{2.3}
\]
\[
\nabla \cdot \mathbf{B} = 0, \tag{2.4}
\]
\[
\nabla \times (\alpha \mathbf{E}) = -\frac{1}{c} \left[ \frac{\partial \mathbf{B}}{\partial t} - (\mathbf{B} \cdot \nabla \omega) \mathbf{m} \right], \tag{2.5}
\]
and
\[
\nabla \times (\alpha \mathbf{B}) = \frac{1}{c} \left[ \frac{\partial \mathbf{E}}{\partial t} - (\mathbf{E} \cdot \nabla \omega) \mathbf{m} \right] + \frac{4\pi \alpha \mathbf{j}}{c}. \tag{2.6}
\]
II. BLACK HOLE ASTROPHYSICS VII

\(\alpha: \) the \textit{lapse function} of the FIDO,

\[
\alpha \equiv \frac{\Delta(\text{proper time } \tau)}{\Delta(\text{coordinate time } t)} = \frac{\rho}{\Sigma} \sqrt{\Delta}.
\]

\(\omega: \) the \textit{angular velocity} of the FIDO,

\[
\omega = \frac{2aMr}{\Sigma^2}.
\]

\(\Omega^F: \) the \textit{angular velocity} of the \textit{magnetic field line}.

\(M: \) the \textit{mass} of a rotating Kerr black hole.

\(a: \) the \textit{angular momentum} per unit mass.

\[
\rho^2 \equiv r^2 + a^2 \cos^2 \theta, \quad \Sigma^2 \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \Delta \equiv r^2 + a^2 - 2Mr.
\]
Define an **m-loop** \( \partial A \), and any **surface** \( A \) bounded by \( \partial A \) which is not intersecting the event horizon of the black hole.

We also define the **outward normal vector** \( dS \) of an infinitesimal area on \( A \).

Define:

\( I \) : the **total electric current** passing downward through \( A \),

\( \Psi \) : the **total magnetic flux** passing upward through \( A \)

\( \Phi \) : the **total electric flux** passing upward through \( A \)
Investigated by Macdonald & Thorne (1982, MT).
II. BLACK HOLE ASTROPHYSICS X

Region 1

The Stationary, Relativistic Grad-Shafranov Equation [Macdonald & Thorne 1982]

\[
\nabla \cdot \left[ \frac{\alpha}{\omega^2} \left\{ 1 - \left( \frac{\omega - \Omega^F}{\alpha c} \right)^2 \right\} \nabla \Psi \right] - \frac{\omega - \Omega^F}{\alpha c^2} \nabla \Omega^F \cdot \nabla \Psi \\
+ \frac{16\pi^2 I}{\alpha c^2 \omega^2} \frac{dI}{d\Psi} = 0
\]

(2.7)

The Grad-Shafranov Equation: A second-order, non-linear partial differential equation that describes the relationship between the plasma flow distribution and pressure in terms of magnetic flux function \( \Psi \).
II. BLACK HOLE ASTROPHYSICS XI

Investigated by Park & Vishniac (1989, PV).

Axisymmetric, Nonstationary

\[ I(t, r) \equiv -\int_A \alpha \mathbf{j} \cdot dS \]

\[ \Psi(t, r) \equiv \int_A \mathbf{B} \cdot dS \]

\[ \Phi(t, r) \equiv \int_A \mathbf{E} \cdot dS \]

\[ E^T = -\frac{2}{\alpha c \omega} \left( \frac{\dot{\Psi}}{4\pi} \right) e_\phi \]

\[ B^T = -\frac{2}{\alpha c \omega} \left( I - \frac{\dot{\Phi}}{4\pi} \right) e_\phi \]

\[ E^P = E^P \]

\[ B^P = \frac{\nabla \Psi \times e_\phi}{2\pi \omega} \]

\[ v^T_F = \frac{\omega - \Omega_F}{\alpha} \omega e_\phi \]

\[ v^P_F = \frac{\omega}{\alpha} e_\phi \]
## II. BLACK HOLE ASTROPHYSICS XII

<table>
<thead>
<tr>
<th></th>
<th>Axisymmetric, Stationary</th>
<th>Axisymmetric, Nonstationary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Electric Current</strong></td>
<td>$I(r) \equiv -\int_A \alpha j \cdot dS$</td>
<td>$I(t, r) \equiv -\int_A \alpha j \cdot dS$</td>
</tr>
<tr>
<td><strong>Magnetic Flux</strong></td>
<td>$\Psi(r) \equiv \int_A B \cdot dS$</td>
<td>$\Psi(t, r) \equiv \int_A B \cdot dS$</td>
</tr>
<tr>
<td><strong>Electric Flux</strong></td>
<td>$\Phi(r) \equiv \int_A E \cdot dS$</td>
<td>$\Phi(t, r) \equiv \int_A E \cdot dS$</td>
</tr>
<tr>
<td><strong>Electric Field (Toroidal)</strong></td>
<td>$E^T = 0$</td>
<td>$E^T = -\frac{2}{a \omega} \left( \frac{\dot{\Psi}}{4\pi} \right) e_\phi$</td>
</tr>
<tr>
<td><strong>Magnetic Field (Toroidal)</strong></td>
<td>$B^T = -\frac{2I}{a \omega} e_\phi$</td>
<td>$B^T = -\frac{2}{a \omega} \left( I - \frac{\dot{\phi}}{4\pi} \right) e_\phi$</td>
</tr>
<tr>
<td><strong>Electric Field (Poloidal)</strong></td>
<td>$E^P = \frac{\nabla \Phi \times e_\phi}{2\pi \omega}$</td>
<td>$E^P = E^P$</td>
</tr>
<tr>
<td><strong>Magnetic Field (Poloidal)</strong></td>
<td>$B^P = \frac{\nabla \Psi \times e_\phi}{2\pi \omega}$</td>
<td>$B^P = \frac{\nabla \Psi \times e_\phi}{2\pi \omega}$</td>
</tr>
</tbody>
</table>
II. BLACK HOLE ASTROPHYSICS XIII

Region 1

Based on PV model, Song & Park induced the nonstationary, relativistic Grad-Shafranov Equation. [Song & Park 2016]

\[
\begin{align*}
\nabla \cdot \left[ \frac{\alpha}{\varpi^2} \left\{ 1 - \left( \frac{\omega - \Omega_F}{\alpha c} \varpi \right)^2 \right\} \nabla \Psi \right] & - \frac{\omega - \Omega_F}{\alpha c^2} \nabla \Omega_F \cdot \nabla \Psi \\
+ \frac{4\pi \dot{\varpi}}{\alpha^2 c^3 \varpi} \left( I - \frac{\dot{\Phi}}{4\pi} \right) \frac{\partial \Omega_F}{\partial z} & + \frac{4\pi}{c^3} \frac{\partial}{\partial z} \left[ \frac{\dot{\varpi}}{\alpha \varpi} \frac{\omega - \Omega_F}{\alpha} \left( I - \frac{\dot{\Phi}}{4\pi} \right) \right] \\
+ \frac{1}{\alpha c^2 \varpi^2} \left[ \left( \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\varpi}}{\varpi} \right) \dot{\Psi} - \ddot{\Psi} \right] & + \frac{\dot{\varphi}}{c^2 \varpi} \frac{\omega - \Omega_F}{\alpha} \frac{\partial \Psi}{\partial \varpi} \\
- \frac{16\pi^2 \xi}{c^2 \varpi^2} \left( I - \frac{\dot{\Phi}}{4\pi} \right) & = 0
\end{align*}
\]
Region 2

The Stationary, Newtonian Grad-Shafranov Equation
(by setting $\alpha = 1$, $\omega = 0$, and $\varpi = R$ in (2.7))
[Goldreich & Julian 1969]

$$\nabla \cdot \left[ \frac{1}{R^2} \left( 1 - \left( \frac{R \Omega_F}{c} \right)^2 \right) \nabla \Psi \right] + \frac{\Omega_F}{c^2} \nabla \Omega_F \cdot \nabla \Psi + \frac{16 \pi^2 I}{c^2 R^2} \frac{dI}{d\Psi} = 0 \quad (2.9)$$

The so-called “Pulsar Equation” investigated by Goldreich & Julian.
II. BLACK HOLE ASTROPHYSICS XV

Region 2

The Nonstationary, Newtonian Grad-Shafranov Equation
(by setting $\alpha = 1$, $\omega = 0$, and $\varpi = R$ in (2.8))
[Song & Park 2016]

\[
\begin{align*}
\nabla \cdot \left[ \frac{1}{R^2} \left\{ 1 - \left( \frac{R\Omega_F}{c} \right)^2 \right\} \nabla \Psi \right] &+ \frac{\Omega_F}{c^2} \nabla \Omega_F \cdot \nabla \Psi \\
+ \frac{4\pi \dot{R}}{c^3 R} \left( I - \frac{\hat{\Phi}}{4\pi} \right) \frac{\partial \Omega_F}{\partial z} &- \frac{4\pi}{c^3} \frac{\partial}{\partial z} \left[ \frac{\dot{R} \Omega_F}{R} \left( I - \frac{\hat{\Phi}}{4\pi} \right) \right] \\
+ \frac{1}{c^2 R^2} \left( \frac{\ddot{R}}{R} \dot{\Psi} - \dot{\Psi} \right) &- \frac{\Omega_F \dot{\varphi}}{c^2 R} \frac{\partial \Psi}{\partial R} - \frac{16\pi^2 \xi}{c^2 R^2} \left( I - \frac{\hat{\Phi}}{4\pi} \right) = 0
\end{align*}
\] (2.10)
II. BLACK HOLE ASTROPHYSICS XVI

Region 3

In Newtonian acceleration region, we can define the Alfvenic Mach number $M_A$ and the entropy $s$ by

$$M_A^2 \equiv 4\pi \rho \kappa^2,$$

$$s \equiv \frac{k_B}{\Gamma - 1},$$

where $\rho$ is the density, $\kappa$ is a scalar function of position, $\Gamma$ is the adiabatic index, and $k_B$ is the Boltzmann’s constant.
Region 3

The Stationary Grad-Shafranov Equation [Okamoto 1975]

\[
\nabla \cdot \left[ \frac{1}{R^2} \left\{ 1 - \left( \frac{R\Omega_F}{c} \right)^2 - M_A^2 \right\} \nabla \Psi \right] + \frac{\Omega_F}{c^2} \nabla \Omega_F \cdot \nabla \Psi \\
+ \frac{1}{R^2} \left( \nabla M_A^2 - \frac{M_A^2}{\kappa} \nabla \kappa \right) \cdot \nabla \Psi + \frac{16\pi^2 I}{c^2 R^2} I' \\
= -\frac{4\pi^2 M_A^2}{\kappa^2} \left[ \varepsilon' - (\Omega_F L)' + R v^T \Omega_F' \right] + \frac{16\pi^3 \eta B^T}{R} (R v^T)' \\
+ \frac{16\pi^3}{k_B} P_s'
\]

(2.12)

where \((\,')\equiv d/d\Psi\).

The so-called “Jet Equation” investigated by Okamoto.
II. BLACK HOLE ASTROPHYSICS XVIII

Region 3

The Nonstationary Grad-Shafranov Equation

[Song & Park 2016]

\[
\nabla \cdot \left[ \frac{1}{R^2} \left\{ 1 - \left( \frac{R \Omega_F}{c} \right)^2 - M_A^2 - \frac{R^2 \Omega_F \dot{C}}{c^2} \frac{\partial \Psi}{\partial z} \right\} \nabla \Psi \right] + \frac{\Omega_F - \dot{C}}{c^2} \nabla \Omega_F \cdot \nabla \Psi + \frac{1}{R^2} \left( \nabla M_A^2 - \frac{M_A^2}{\kappa} \nabla \kappa \right) \cdot \nabla \Psi + \frac{1}{c^2 R^2} \nabla \left( R^2 \Omega_F \dot{C} \frac{\partial \Psi}{\partial z} \right) \cdot \nabla \Psi - \frac{2 \Omega_F \dot{C}}{c^2 R} \frac{\partial \Psi}{\partial R} + \frac{1}{c^2 R^2} \left( \frac{\dot{R}}{R} - \dot{\Psi} - R \Omega_F \dot{\phi} \frac{\partial \Psi}{\partial R} \right) - \frac{4\pi}{c^3 R} \left( \Omega_F + \dot{C} \right) \frac{\partial}{\partial z} \left[ \dot{R} \left( I - \frac{\dot{\phi}}{4\pi} \right) \right]
\]

\[
+ \frac{16\pi^2}{c^2 R^2 |\nabla \Psi|^2} \left( I - \frac{\dot{\phi}}{4\pi} \right) \nabla I \cdot \nabla \Psi
\]

\[
= -\frac{4\pi^2 M_A^2}{\kappa^2 |\nabla \Psi|^2} \left[ \nabla \epsilon - \nabla (\Omega_F L) + R v_T \nabla \Omega_F + \frac{2\kappa}{c R^2} \left( I - \frac{\dot{\phi}}{4\pi} \right) \nabla (R v_T) \right] \cdot \nabla \Psi + \frac{4\pi^2 M_A^2}{k_B \rho \kappa^2 |\nabla \Psi|^2} \left[ P \nabla s + s \nabla P - \frac{k_B P}{\rho} \left( 1 + \frac{s}{k_B} \right) \nabla \rho \right] \cdot \nabla \Psi
\]

\[
- \frac{4\pi^2 M_A^2}{\kappa^2 |\nabla \Psi|^2} \left[ \frac{1}{\rho} \left( \dot{\psi}_R - v_T \dot{\phi} \right) \frac{\partial \Psi}{\partial R} + \frac{\dot{\psi}_z}{\rho} \frac{\partial \Psi}{\partial z} + |\nabla \Psi| H \right]
\]

(2.13)
where

\[ \dot{C} \equiv \frac{\dot{R} M_A^2}{c \rho \kappa^2 R |\nabla \Psi|^2} \left( I - \frac{\dot{\Phi}}{4\pi} \right), \quad (2.14) \]

and

\[
\dot{H} \equiv \frac{\kappa}{2\pi R} \left[ \dot{R} D_\perp \left( \frac{\partial \Psi}{\partial z} \right) + \frac{1}{|\nabla \Psi|} \frac{\partial \Psi}{\partial R} \left\{ \frac{\partial \dot{\Psi}}{\partial z} - \frac{\partial \dot{R}}{\partial z} \frac{\partial \Psi}{\partial R} + \frac{\partial \dot{R}}{\partial R} \frac{\partial \Psi}{\partial z} \right\} \right] \\
+ \frac{\kappa}{2\pi R} \frac{\partial \Psi}{\partial z} \left[ D_\perp \dot{R} + \frac{\dot{\Psi}}{R |\nabla \Psi|} - \frac{1}{|\nabla \Psi|} \frac{\partial \dot{\Psi}}{\partial R} + \frac{\dot{R}}{\kappa} D_\perp \kappa \right] \\
+ \dot{R} D_\perp \dot{R} + \frac{\dot{R}}{|\nabla \Psi|} \frac{\partial \Psi}{\partial R}. \]

(2.15)
### II. BLACK HOLE ASTROPHYSICS XVI

<table>
<thead>
<tr>
<th>Regions</th>
<th>Physics</th>
<th>The Stationary GSEs</th>
<th>The Nonstationary GSEs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Region 1</td>
<td>$0 &lt; \alpha &lt; 1$, $0 &lt; \omega &lt; \Omega_H$, $M_A^2 = 0$, $s = 0$</td>
<td>(2.7)</td>
<td>(2.8)</td>
</tr>
<tr>
<td>Region 2</td>
<td>$\varpi = R$, $\alpha = 1$, $\omega = 0$, $M_A^2 = 0$, $s = 0$</td>
<td>(2.9)</td>
<td>(2.10)</td>
</tr>
<tr>
<td>Region 3</td>
<td>$\varpi = R$, $\alpha = 1$, $\omega = 0$, $M_A^2 \neq 0$, $s \neq 0$</td>
<td>(2.12)</td>
<td>(2.13)</td>
</tr>
</tbody>
</table>

**Table:** The Grad-Shafranov Equations (GSEs)
IV

CONCLUSION
IV. CONCLUSION

Evidently, there are many nonstationary phenomena beyond the scope of my paper that can play important roles in a realistic picture of a black hole magnetosphere.

It is unfortunate that the GSEs for the nonstationary case in Regions 1, 2, and 3, (2.8), (2.10), and (2.13), seem to be too complicated to solve directly via analytic methods. If we can solve these GSEs numerically, then we will understand the axisymmetric, nonstationary black hole magnetosphere in more rigorous ways.

My future work may include the luminosity variability, the precession of the central black holes, the formation of the nodes in the middle of the astrophysical jets, and so on.
CONCLUSION


Thank You!
The Lapse Function

For non-inertial observers, and in general relativity, coordinate systems can be chosen more freely. For a clock whose spatial coordinates are constant, the relationship between proper time $\tau$ and coordinate time $t$, i.e. **the rate of time dilation**, is given by

$$\frac{d\tau}{dt} = \sqrt{-g_{00}}$$

where $g_{00}$ is a component of the metric tensor, which incorporates gravitational time dilation (under the convention that the zeroth component is timelike).
In the theory of relativity, it is convenient to express results in terms of a spacetime coordinate system relative to an implied observer. In many (but not all) coordinate systems, an event is specified by one time coordinate and three spatial coordinates. The time specified by the time coordinate is referred to as **coordinate time** to distinguish it from proper time.
In relativity, **proper time** along a timelike world line is defined as the time as measured by a clock following that line. It is thus independent of coordinates, and a Lorentz scalar. This is the quantity of interest, since proper time itself is fixed only up to an arbitrary additive constant, namely the setting of the clock at some event along the world line. The proper time between two events depends not only on the events but also the world line connecting them, and hence on the motion of the clock between the events. It is expressed as an integral over the world line. An accelerated clock will measure a smaller elapsed time between two events than that measured by a non-accelerated (inertial) clock between the same two events. The twin paradox is an example of this effect.
A **force-free** magnetic field is a magnetic field that arises when the plasma pressure is so small, relative to the magnetic pressure, that the plasma pressure may be ignored, and so only the magnetic pressure is considered. For a force free field, the electric current density is either zero or parallel to the magnetic field. The name "**force-free**" comes from being able to neglect the force from the plasma.